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Three-dimensional Lorentz manifolds admitting a parallel null vector field

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Abstract
Curvature properties of three-dimensional Lorentz manifolds admitting a parallel degenerate line field are examined. A complete characterization of those manifolds being locally symmetric or locally conformally flat is obtained. The results of this study show nice families of examples of such properties within the Lorentzian setting.

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1. Introduction

It is well known that the existence of a parallel line field on a Riemannian manifold gives rise to a local decomposition of the manifold as a direct product. This property extends to semi-Riemannian manifolds whenever the line field is nondegenerate, i.e., is spanned by a non-null locally defined vector field. However, the geometrical consequences of the existence of a parallel degenerate line field are not yet well understood.

An additional motivation for investigating the influence of a parallel null vector field comes from the nonuniqueness of the Levi-Civita connection in the semi-Riemannian setting. Indeed, two non-homothetic Riemannian metrics give rise to the same Levi-Civita connection if and only if the manifold decomposes locally as a product. Moreover non-locally decomposable different Lorentzian metrics may have the same Levi-Civita connection if and only if there exists a parallel null vector on the manifold [8].

Recall that a parallel degenerate line field $\mathcal{D}$ (i.e., $\nabla\mathcal{D} \subset \mathcal{D}$) is not necessarily spanned by a locally defined parallel null vector U but instead $\nabla U = \omega \otimes U$. Such vector fields are usually called recurrent vector fields in the literature (cf [4, 5] and the references therein for more information). Note that in such case $\|\nabla U\|^2 = 0$ but $\nabla U \neq 0$.

In this paper we will focus on the curvature properties of three-dimensional Lorentz manifolds admitting a parallel degenerate line field. Since the curvature tensor of any three-manifold is completely determined by its Ricci tensor, we pay special attention to the behaviour of the Ricci operator.

2. The geometry of a three-dimensional Lorentz manifold

An important observation for our purpose is the existence of canonical coordinates adapted to a parallel plane field (cf [10, 11]). Hence, a three-dimensional Lorentz manifold (M, g) admitting a parallel degenerate line field has local coordinates (t, x, y) where the Lorentzian metric tensor is expressed

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(t,x,y) \end{pmatrix} \tag{1}$$

for some function $f(t, x, y)$, where $\varepsilon = \pm 1$ and the parallel degenerate line field becomes $\mathcal{D} = \left\langle \frac{\partial}{\partial t} \right\rangle$.

Moreover, note that the existence of a parallel null vector $U = \partial/\partial t$ influences the coordinates above in the sense that the function $f(t, x, y) \equiv f(x, y)$ [10].

2.1. Levi-Civita connection

It follows after a straightforward calculation that the Levi-Civita connection of any metric (1) is given by

$$\begin{aligned} \nabla_{\partial_t} \partial_y &= \frac{1}{2} f_t \partial_t \\ \nabla_{\partial_x} \partial_y &= \frac{1}{2} f_x \partial_t \\ \nabla_{\partial_y} \partial_y &= \frac{1}{2}(f f_t + f_y)\partial_t - \frac{1}{2\varepsilon} f_x \partial_x - \frac{1}{2} f_t \partial_y, \end{aligned} \tag{2}$$

where $\partial_t, \partial_x, \partial_y$ are the coordinate vector fields $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, respectively. Hence, if (M, g) admits a parallel null vector field, then the associated Levi-Civita connection satisfies

$$\nabla_{\partial_x} \partial_y = \frac{1}{2} f_x \partial_t, \qquad \nabla_{\partial_y} \partial_y = \frac{1}{2} f_y \partial_t - \frac{1}{2\varepsilon} f_x \partial_x. \tag{3}$$

Deciding on the geodesic completeness of a semi-Riemannian metric is not an easy task. However, a simple criterion for geodesic completeness is as follows [6]: 'A semi-Riemannian metric defined globally on $\mathbb{R}^n$ whose Christoffel symbols satisfy $\Gamma_{ij}^k = 0$ for all $i, j < k$ is geodesically complete'.

Hence, *any metric* (1) *on* $\mathbb{R}^3$ *with* $U = \partial/\partial t$ *a parallel null vector is geodesically complete*.

2.2. Curvature tensor

Let R denote the curvature tensor taken with the sign convention $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. Then the nonzero components of the curvature tensor of any metric (1) are given by

$$R(\partial_t, \partial_y)\partial_t = -\frac{1}{2} f_{tt} \partial_t$$

$$R(\partial_t, \partial_y)\partial_x = -\frac{1}{2} f_{tx} \partial_t$$

$$
\begin{aligned}
R(\partial_t, \partial_y)\partial_y &= -\frac{1}{2} f f_{tt} \partial_t + \frac{1}{2\varepsilon} f_{tx} \partial_x + \frac{1}{2} f_{tt} \partial_y \\
R(\partial_x, \partial_y)\partial_t &= -\frac{1}{2} f_{tx} \partial_t \\
R(\partial_x, \partial_y)\partial_x &= -\frac{1}{2} f_{xx} \partial_t \\
R(\partial_x, \partial_y)\partial_y &= -\frac{1}{2} f f_{tx} \partial_t + \frac{1}{2\varepsilon} f_{xx} \partial_x + \frac{1}{2} f_{tx} \partial_y .
\end{aligned} \tag{4}
$$

Further note that the existence of parallel null vector field simplifies (4) as follows:

$$
R(\partial_x, \partial_y)\partial_x = -\frac{1}{2} f_{xx} \partial_t, \qquad R(\partial_x, \partial_y)\partial_y = \frac{1}{2\varepsilon} f_{xx} \partial_x . \tag{5}
$$

As a matter of notation, let Ric and Sc be the Ricci tensor and the scalar curvature of (M, g), defined by $\mathrm{Ric}(X, Y) = \mathrm{trace}\{Z \mapsto R(X, Z)Y\}$ and $\mathrm{Sc} = \mathrm{trace}\,\mathrm{Ric}$, respectively. Moreover, let $\widehat{\mathrm{Ric}}$ be the Ricci operator defined by $\langle \widehat{\mathrm{Ric}}(X), Y\rangle = \mathrm{Ric}(X, Y)$. Then the Ricci tensor of any metric (1) satisfies

$$
\mathrm{Ric} = \begin{pmatrix} 0 & 0 & \frac{1}{2} f_{tt} \\ 0 & 0 & \frac{1}{2} f_{tx} \\ \frac{1}{2} f_{tt} & \frac{1}{2} f_{tx} & \frac{1}{2\varepsilon}(\varepsilon f f_{tt} - f_{xx}) \end{pmatrix}, \tag{6}
$$

when expressed in the local coordinate basis. Moreover, the Ricci operator $\widehat{\mathrm{Ric}}$ of a metric (1), when expressed in the coordinate basis, takes the form

$$
\widehat{\mathrm{Ric}} = \begin{pmatrix} \frac{1}{2} f_{tt} & \frac{1}{2} f_{tx} & -\frac{1}{2\varepsilon} f_{xx} \\ 0 & 0 & \frac{1}{2\varepsilon} f_{tx} \\ 0 & 0 & \frac{1}{2} f_{tt} \end{pmatrix}. \tag{7}
$$

Hence, the Ricci operator has eigenvalues

$$
\lambda_1 = 0, \qquad \lambda_2 = \lambda_3 = \tfrac{1}{2} f_{tt},
$$

and thus the scalar curvature satisfies

$$
\mathrm{Sc} = f_{tt}. \tag{8}
$$

Observe from the eigenvalue structure of $\widehat{\mathrm{Ric}}$ that a metric (1) is Einstein if and only if it is Ricci flat (indeed, is flat since $\dim M = 3$).

Remark 1. Examples of Lorentzian three-manifolds with constant Ricci curvatures are easily constructed from (7). Indeed, *a metric* (1) *has constant Ricci eigenvalues if and only if it is locally given by*

$$
f(t, x, y) = \kappa t^2 + t P(x, y) + \xi(x, y),
$$

for any functions $P(x, y), \xi(x, y)$. Further note that even in this case the Ricci operator is not necessarily diagonalizable. In fact, it follows from (7) that $\widehat{\mathrm{Ric}}$ is diagonalizable with respect to an orthonormal basis if and only if $f_{tx} = 0$ and $f_{xx} = 0$. Hence, *a metric* (1) *with constant Ricci curvatures has diagonalizable Ricci operator if and only if*

$$
f(t, x, y) = \kappa t^2 + t P(y) + x\eta(y) + \xi(y)
$$

for any functions P, η and ξ.

Remark 2. A scalar curvature invariant of order m is a polynomial in the components $\nabla^k_{i_1,\ldots,i_k} R_{abcd}$, $k = 0, \ldots$ of the curvature tensor and its m covariant derivatives with respect to some orthonormal basis, which is independent of the choice of this orthonormal basis at each point $p \in M$.

Let (M, g) be a three-dimensional Lorentz manifold admitting a parallel null vector field. Then, the nonzero components of the curvature tensor are those given by

$$R(\partial_x, \partial_y)\partial_x = -\frac{1}{2} f_{xx}\partial_t, \qquad R(\partial_x, \partial_y)\partial_y = \frac{1}{2\varepsilon} f_{xx}\partial_x.$$

Moreover, the covariant derivative of the curvature tensor is given by

$$\big(\nabla_{\partial_x} R\big)(\partial_x, \partial_y)\partial_x = -\tfrac{1}{2} f_{xxx}\partial_t, \qquad \big(\nabla_{\partial_y} R\big)(\partial_x, \partial_y)\partial_x = -\tfrac{1}{2} f_{xxy}\partial_t.$$

Hence, since $U = \partial_t$ is parallel, it follows that the nonzero components of the higher order covariant derivatives of the curvature tensor produce higher order derivatives of f in the direction of ∂_t. Moreover, since the inverse of the metric (1) satisfies

$$g^{-1}_{(t,x,y)} = \begin{pmatrix} -f(x,y) & 0 & 1 \\ 0 & \frac{1}{\varepsilon} & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{9}$$

it follows that $(g^{-1})(\partial_x, \partial_y) = 0$, and thus *all scalar curvature invariants of a three-dimensional Lorentz manifold* (M, g) *admitting a parallel null vector field vanish identically*.

3. Einstein-like manifolds

Einstein metrics as well as constant scalar curvature metrics are important classes of semi-Riemannian manifolds. However, in addition to parallel Ricci tensor (which lies between both classes above) there are other Ricci curvature properties which deserve further investigation.

Definition 3. *Following the terminology of Gray [7] we say that*

- (M, g) *has cyclic parallel Ricci tensor if* $(\nabla_X \mathrm{Ric})(X, X) = 0$ *for all vectors* X.
- *The Ricci tensor is Codazzi (or equivalently,* (M, g) *has harmonic curvature) if* $(\nabla_X \mathrm{Ric})(Y, Z) = (\nabla_Y \mathrm{Ric})(X, Z)$ *for all* X, Y, Z.
- (M, g) *is said to be a* $\mathcal{C}^\perp$*-manifold if*

$$(\nabla_X \mathrm{Ric})(Y, Z) = \frac{1}{(n+2)(n-1)} \Big\{ nX(Sc)\langle Y, Z\rangle + \frac{1}{2}(n-2)[Y(Sc)\langle X, Z\rangle + Z(Sc)\langle X, Y\rangle] \Big\}$$

for all vector fields X, Y, Z *on* M.

Theorem 4. *A three-dimensional Lorentz manifold admitting a parallel degenerate line field is locally symmetric if and only if it the function* f *in the metric (1) takes either form of the following:*

$$f(t, x, y) = t^2\kappa + t(xP(y) + Q(y)) + \frac{x^2}{4\kappa} P(y)^2 + x\beta(y) + \xi(y), \tag{10}$$

for any functions $P(y)$, $Q(y)$, $\beta(y)$ *and* $\xi(y)$ *satisfying the linear differential equation*

$$P' + \tfrac{1}{2} PQ = \kappa\beta$$

and any real constant $\kappa \neq 0$, *or*

$$f(t, x, y) = tQ(y) + x^2\alpha(y) + x\beta(y) + \xi(y), \tag{11}$$

for any functions $Q(y)$, $\alpha(y)$, $\beta(y)$ and $\xi(y)$ satisfying the linear differential equation

$$\alpha' + Q\alpha = 0.$$

Remark 5. The metrics given by (10) have scalar curvature $Sc = 2\kappa$ while those defined by (11) have zero scalar curvature.

Proof of theorem 4. Note that, since the curvature tensor of a three-dimensional manifold is completely determined by the Ricci tensor, the metric is locally symmetric if and only if the Ricci tensor is parallel. The covariant derivative of the Ricci tensor is obtained after some calculation from (2) and (6), showing that the nonzero components are given by

$$\begin{aligned}
\big(\nabla_{\partial_t}\mathrm{Ric}\big)(\partial_t, \partial_y) &= \frac{1}{2} f_{ttt}\\
\big(\nabla_{\partial_t}\mathrm{Ric}\big)(\partial_x, \partial_y) &= \big(\nabla_{\partial_x}\mathrm{Ric}\big)(\partial_t, \partial_y) = \frac{1}{2} f_{xtt}\\
\big(\nabla_{\partial_t}\mathrm{Ric}\big)(\partial_y, \partial_y) &= -\frac{1}{2\varepsilon}(f_{xxt} - \varepsilon f f_{ttt})\\
\big(\nabla_{\partial_x}\mathrm{Ric}\big)(\partial_x, \partial_y) &= \frac{1}{2} f_{xxt}\\
\big(\nabla_{\partial_x}\mathrm{Ric}\big)(\partial_y, \partial_y) &= -\frac{1}{2\varepsilon}(f_{xxx} - \varepsilon f f_{xtt})\\
\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_t, \partial_y) &= \frac{1}{2} f_{ytt}\\
\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_x, \partial_y) &= \frac{1}{4}(f_t f_{tx} + 2 f_{txy} - f_x f_{tt})\\
\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_y, \partial_y) &= -\frac{1}{2\varepsilon}(f_{yxx} + f_{xx} f_t - f_x f_{tx} - \varepsilon f f_{ytt}).
\end{aligned} \tag{12}$$

Now, it follows from the first condition in (12) that the defining function $f(t, x, y)$ satisfies

$$f(t, x, y) = t^2\kappa(x, y) + t\lambda(x, y) + \xi(x, y),$$

and from the components $\big(\nabla_{\partial_t}\mathrm{Ric}\big)(\partial_x, \partial_y)$ and $\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_t, \partial_y)$ one has that $\kappa(x, y)$ is constant $\kappa(x, y) \equiv \kappa$.

Next, since $\big(\nabla_{\partial_x}\mathrm{Ric}\big)(\partial_x, \partial_y) = \frac{1}{2} f_{xxt} = 0$ one has that $\lambda_{xx} = 0$, and thus

$$f(t, x, y) = t^2\kappa + t(xP(y) + Q(y)) + \xi(x, y),$$

for some functions P, Q, ξ. Moreover, since $f_{xtt} = 0$ one has $\big(\nabla_{\partial_x}\mathrm{Ric}\big)(\partial_y, \partial_y) = -\frac{1}{2\varepsilon}(f_{xxx} - \varepsilon f f_{xtt}) = -\frac{1}{2\varepsilon} f_{xxx} = 0$, and thus $\xi_{xxx} = 0$, which shows that

$$f(t, x, y) = t^2\kappa + t(xP(y) + Q(y)) + x^2\alpha(y) + x\beta(y) + \xi(y).$$

Next, note that a metric (1) defined by a function $f(t, x, y)$ as above has parallel Ricci tensor if and only if

$$\begin{aligned}
\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_x, \partial_y) &= \frac{1}{4}(f_t f_{tx} + 2 f_{txy} - f_x f_{tt})\\
&= \frac{1}{4}(xP^2 + PQ - 4x\kappa\alpha - 2\kappa\beta + 2P') \equiv 0,\\
\big(\nabla_{\partial_y} Ric\big)(\partial_y, \partial_y) &= -\frac{1}{2\varepsilon}(f_{yxx} + f_{xx} f_t - f_x f_{tx} - \varepsilon f f_{ytt})\\
&= \frac{1}{2\varepsilon}(tP^2 + P\beta - 2(2t\kappa + Q)\alpha + \alpha') \equiv 0.
\end{aligned}$$

Finally we will consider separately the different possibilities $\kappa \neq 0$ and $\kappa = 0$.

If $\kappa = 0$, then the first equation above gives $P(y) \equiv 0$, and thus the second one becomes

$$\alpha'(y) + Q(y)\alpha(y) \equiv 0.$$

This shows that a metric (1) given by

$$f(t, x, y) = tQ(y) + x^2\alpha(y) + x\beta(y) + \xi(y)$$

is locally symmetric if and only if $\alpha(y)$ is a solution of the linear differential equation $\alpha'(y) + Q(y)\alpha(y) = 0$.

If $\kappa \neq 0$, we have the necessary and sufficient condition as a set of PDEs as follows:

$$P^2 - 4\kappa\alpha = 0, \qquad PQ - 2\kappa\beta + 2P' = 0, \qquad P\beta - 2Q\alpha - 2\alpha' = 0,$$

where the third equation is a consequence of the second one (just using that $\alpha = \frac{1}{4\kappa}P^2$). Hence, a metric (1) is locally symmetric if and only if the defining function $f(t, x, y)$ satisfies

$$f(t, x, y) = t^2\kappa + t(xP(y) + Q(y)) + \frac{x^2}{4\kappa}P(y)^2 + x\beta(y) + \xi(y),$$

where P is a solution of the linear differential equation

$$P' + \tfrac{1}{2}PQ = \kappa\beta.$$

□

The Ricci operator of a locally symmetric metric tensor (1) defined by a function f satisfying (10), when expressed in the coordinate vector fields satisfies

$$\widehat{\mathrm{Ric}} = \begin{pmatrix} \kappa & \frac{1}{2}P(y) & -\frac{1}{4\kappa\varepsilon}P(y)^2 \\ 0 & 0 & \frac{1}{2\varepsilon}P(y) \\ 0 & 0 & \kappa \end{pmatrix},$$

which is diagonalizable with respect to an orthonormal basis if and only if $P \equiv 0$. Note that for any metric tensor (1) defined by (10) with $P \equiv 0$ one has

$$f(t, x, y) = t^2\kappa + tQ(y) + \xi(y),$$

so it is locally a product of a Lorentz surface of constant Gaussian curvature κ (defined by the coordinates (t, y)) and an interval.

The Ricci operator of a locally symmetric metric tensor (1) defined by a function f given by (11), when expressed in the coordinate vector fields satisfies

$$\widehat{\mathrm{Ric}} = \begin{pmatrix} 0 & 0 & -\frac{1}{\varepsilon}\alpha(y) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is diagonalizable with respect to an orthonormal basis if and only if $\alpha \equiv 0$ (if and only if the metric is flat).

Theorem 6. *A three-dimensional Lorentz manifold admitting a parallel null vector field is locally symmetric if and only if it is locally given by (1) where the function f satisfies*

$$f(x, y) = x^2\alpha + x\beta(y) + \xi(y), \tag{13}$$

for any functions $\beta(y)$ and $\xi(y)$ and any constant $\alpha \in \mathbb{R}$.

3.1. Cyclic-parallel Ricci tensor and $\mathcal{C}^{\perp}$-manifolds

Clearly any manifold with parallel Ricci tensor has cyclic-parallel Ricci tensor and it is a $\mathcal{C}^{\perp}$-manifold, but the converse is not true in general.

Theorem 7. *Let (M, g) be a three-dimensional Lorentz manifold admitting a parallel degenerate line field. Then the Ricci tensor of M is cyclic parallel if and only if it is parallel.*

Proof. Let σ denote the cyclic sum. Then for metric (1), the nonzero components of

$$\sigma(\nabla_X \mathrm{Ric})(Y, Z) = (\nabla_X \mathrm{Ric})(Y, Z) + (\nabla_Y \mathrm{Ric})(Z, X) + (\nabla_Z \mathrm{Ric})(X, Y),$$

are given by (compare with (12))

$$\begin{aligned}
\sigma\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_t, \partial_t) &= f_{ttt}\\
\sigma\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_t, \partial_x) &= f_{xtt}\\
\sigma\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_t, \partial_y) &= -\frac{1}{2\varepsilon} f_{xxt} + f_{ytt} + \frac{1}{2} f f_{ttt}\\
\sigma\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_x, \partial_x) &= f_{xxt}\\
\sigma\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_x, \partial_y) &= \frac{1}{2\varepsilon}\{-f_{xxx} + \varepsilon(f_t f_{tx} + 2 f_{txy} - f_x f_{tt} + f f_{xtt})\}\\
\sigma\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_y, \partial_y) &= \frac{1}{2\varepsilon}\{3(-f_{yxx} - f_{xx} f_t + f_x f_{tx} + \varepsilon f f_{ytt})\}.
\end{aligned} \tag{14}$$

Proceeding as in the proof of theorem 4 it follows that the defining function $f(t, x, y)$ of any metric (1) with cyclic parallel Ricci tensor satisfies

$$f(t, x, y) = t^2\kappa + t(xP(y) + Q(y)) + x^2\alpha(y) + x\beta(y) + \xi(y).$$

Now, the necessary and sufficient conditions for $(\nabla_X \mathrm{Ric})(X, X) = 0$ become

$$\sigma\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_x, \partial_y) = \frac{1}{4}(xP^2 + PQ - 4x\kappa\alpha - 2\kappa\beta + 2P') \equiv 0,$$

$$\sigma\big(\nabla_{\partial_y}\mathrm{Ric}\big)(\partial_y, \partial_y) = \frac{1}{2\varepsilon}(tP^2 + P\beta - 2(2t\kappa + Q)\alpha + \alpha') \equiv 0,$$

which shows that the Ricci tensor is parallel (cf the proof of theorem 4). □

Theorem 8. *Let (M, g) be a three-dimensional Lorentz manifold admitting a parallel degenerate line field. Then (M, g) is a $\mathcal{C}^{\perp}$-manifold if and only if the Ricci tensor of M is parallel.*

Proof. Let $\mathcal{C}^{\perp}$ the (0, 3)-tensor field defined by

$$\mathcal{C}^{\perp}(X, Y, Z) = (\nabla_X \mathrm{Ric})(Y, Z) - \frac{1}{(n+2)(n-1)}\Big\{nX(\mathrm{Sc})\langle Y, Z\rangle + \frac{1}{2}(n-2)[Y(\mathrm{Sc})\langle X, Z\rangle + Z(\mathrm{Sc})\langle X, Y\rangle]\Big\}.$$

Now, it follows after a straightforward calculation that

$$\begin{aligned}
\mathcal{C}^{\perp}(\partial_y, \partial_t, \partial_t) &= -\tfrac{1}{10} f_{ttt}\\
\mathcal{C}^{\perp}(\partial_y, \partial_x, \partial_t) &= -\tfrac{1}{20} f_{xtt}\\
\mathcal{C}^{\perp}(\partial_y, \partial_x, \partial_x) &= -\tfrac{3}{10} f_{ytt}
\end{aligned}$$

which shows that the metric (1) is defined by a function

$$f(t,x,y) = t^2\kappa + \xi(x,y)$$

for some constant $\kappa \in \mathbb{R}$. Now, since $\mathcal{C}^{\perp}(\partial_y, \partial_x, \partial_y) = -\frac{1}{2}\kappa\xi_x$ it follows that either $\xi(x,y) \equiv \xi(y)$ (if $\kappa \neq 0$), or otherwise if $\kappa = 0$, then $\mathcal{C}^{\perp}(\partial_y, \partial_x, \partial_x) = -\frac{1}{2\varepsilon}\xi_{xxx}$ and $\mathcal{C}^{\perp}(\partial_y, \partial_y, \partial_y) = -\frac{1}{2\varepsilon}\xi_{yxx}$. Note that in both cases one has that the Ricci tensor is parallel just comparing with the result of theorem 4. □

3.2. Harmonic curvature and local conformal flatness

A three-dimensional manifold is locally conformally flat if and only if the Schouten tensor c vanishes identically, where

$$c(X,Y,Z) = (\nabla_X \mathrm{Ric})(Y,Z) - (\nabla_Y \mathrm{Ric})(X,Z) - \frac{1}{2(n-2)}\{\langle(\nabla_X \mathrm{Sc})Y, Z\rangle - \langle(\nabla_Y \mathrm{Sc})X, Z\rangle\},$$

for all vector fields X, Y, Z. Hence, the Ricci tensor of any locally conformally flat manifold with constant scalar curvature is Codazzi. Now, the nonzero components of the Schouten tensor of a three-dimensional Lorentz metric admitting a parallel degenerate line field are as follows:

$$\begin{aligned}
c(\partial_x, \partial_t, \partial_x) &= -c(\partial_t, \partial_x, \partial_x) = \frac{1}{2}\varepsilon f_{ttt} \\
c(\partial_y, \partial_t, \partial_x) &= -c(\partial_t, \partial_x, \partial_y) = c(\partial_x, \partial_t, \partial_y) = -c(\partial_t, \partial_y, \partial_x) = -\frac{1}{2}f_{xtt} \\
c(\partial_y, \partial_t, \partial_y) &= -c(\partial_t, \partial_y, \partial_y) = \frac{1}{2\varepsilon}f_{xxt} \\
c(\partial_y, \partial_x, \partial_x) &= -c(\partial_x, \partial_y, \partial_x) = -\frac{1}{2}(f_{xxt} + \varepsilon f_{ytt}) \\
c(\partial_y, \partial_x, \partial_y) &= -c(\partial_x, \partial_y, \partial_y) = \frac{1}{4\varepsilon}(2f_{xxx} + \varepsilon f_t f_{xt} + 2\varepsilon f_{txy} - \varepsilon f_x f_{tt}).
\end{aligned} \tag{15}$$

Hence it follows that a metric (1) is locally conformally flat if and only if the defining function $f(t,x,y)$ satisfies

$$f(t,x,y) = t^2\kappa + t(xP(y) + Q(y)) + \xi(x,y) \tag{16}$$

for any functions P, Q and ξ satisfying

$$c(\partial_y, \partial_x, \partial_y) = \frac{1}{4\varepsilon}\{\varepsilon x P^2 + \varepsilon P Q + 2\varepsilon P' - 2\varepsilon\kappa\xi_x + 2\xi_{xxx}\}. \tag{17}$$

Theorem 9. *Let (M,g) be a three-dimensional Lorentz manifold admitting a parallel degenerate line field. Then the following are equivalent:*

(i) The Ricci tensor is Codazzi.
(ii) (M,g) is locally conformally flat.
(iii) There is a local system of coordinates where the metric is given by (1) for a defining function

$$f(t,x,y) = t^2\kappa + t(xP(y) + Q(y)) + \xi(x,y)$$

for arbitrary functions P and Q, where ξ is a solution of the linear differential equation

$$2\xi_{xx} - 2\varepsilon\kappa\xi = \gamma - \frac{x^2}{2}\varepsilon P^2 - \varepsilon x(PQ + 2P')$$

and $\gamma(y)$ is an arbitrary function.

Proof. Note from (16) that any locally conformally flat metric (1) has constant scalar curvature $\mathrm{Sc} = 2\kappa$. Hence the Ricci tensor of any locally conformally flat three-dimensional Lorentz manifold admitting a parallel degenerate line field is Codazzi. Moreover, since any manifold with harmonic curvature has constant scalar curvature, it follows that (i) and (ii) are equivalent.

Next we will obtain the local form of any locally conformally flat metric (1). Note from (17) that $\varepsilon x P^2 + \varepsilon P Q + 2\varepsilon P' - 2\varepsilon\kappa\xi_x + 2\xi_{xxx} = 0$, and thus $2\xi_{xx} - 2\varepsilon\kappa\xi + \varepsilon\frac{x^2}{2}P^2 + x(\varepsilon P Q + 2\varepsilon P') = \gamma$ for some function $\gamma(y)$. □

Remark 10. It follows from the previous theorem that locally conformally flat metrics (1) with vanishing scalar curvature (i.e., $\kappa = 0$) are those given by

$$f(t, x, y) = t(xP + Q) - \varepsilon\frac{x^4}{24}P^2 - \varepsilon\frac{x^3}{6}(PQ + 2P') + \frac{x^2}{2}\gamma + x\delta + \xi,$$

for arbitrary functions $P(y)$, $Q(y)$, $\gamma(y)$, $\delta(y)$ and $\xi(y)$.

Theorem 11. *Let (M, g) be a three-dimensional Lorentz manifold admitting a parallel null vector. Then the following are equivalent:*

(i) The Ricci tensor is Codazzi.
(ii) (M, g) is locally conformally flat.
(iii) There is a local system of coordinates where the metric is given by (1) for a defining function

$$f(t, x, y) = x^2\gamma(y) + x\eta(y) + \xi(y)$$

where γ, η and ξ are arbitrary functions.

Remark 12. The metrics in theorem 11 are locally symmetric if and only if $\gamma(y)$ is a constant function.

The Ricci operator of a locally conformally flat metric as in theorem 9, expresses in the coordinate basis as

$$\widehat{\mathrm{Ric}} = \begin{pmatrix} \kappa & \frac{1}{2}P(y) & -\frac{1}{2\varepsilon}\xi_{xx}(x, y) \\ 0 & 0 & \frac{1}{2\varepsilon}P(y) \\ 0 & 0 & \kappa \end{pmatrix}.$$

Hence, the Ricci operator is diagonalizable with respect to an orthonormal basis if and only if $P \equiv 0$, $\xi_{xx} \equiv 0$.

It was shown in [3] that a complete connected locally conformally flat three-dimensional Riemannian manifold with both $\mathrm{Sc} \geqslant 0$ and $\|\mathrm{Ric}\|^2 = \mathit{constant}$ is either isometric to a space form of constant sectional curvature or to a Riemannian product $M^2(c) \times \mathbb{R}$, or $M^2(c) \times S^1$.

An immediate application of theorem 11 shows that an analogous statement is no longer true in the Lorentzian setting, since any metric (1) with $f(t, x, y) = x^2\gamma(y) + x\eta(y) + \xi(y)$ for arbitrary functions γ, η and ξ is locally conformally flat with $\mathrm{Sc} = 0$ and $\|\mathrm{Ric}\|^2 = 0$ (see expression of the inverse of the metric (9)). Moreover, any such metric defined on $\mathbb{R}^3$ is geodesically complete.

4. Summary and conclusions

After stating some basic facts in section 2, we considered some different Einstein-like properties [7] showing that

- the Ricci tensor of a three-dimensional Lorentz manifold admitting a parallel degenerate line field is cyclic parallel if and only if it is parallel, and hence locally symmetric (cf theorem 8).
- A three-dimensional Lorentz manifold admitting a parallel degenerate line field has harmonic curvature (or equivalently the Ricci tensor is Codazzi) if and only if it is locally conformally flat (cf theorem 9).

As an application, we showed a family of complete locally conformally flat Lorentz manifolds with zero scalar curvature which are neither of constant sectional curvature nor isometric to a product of a surface of constant curvature and a real line.

It is worth mentioning that any three-dimensional Lorentz manifold admitting a parallel null vector field has vanishing scalar curvature invariants while it need not be locally homogeneous (see also [2]). Moreover, note that many such examples can be constructed to be geodesically complete and locally conformally flat but not locally homogeneous.

Finally, note that those locally conformally flat metrics predicted in [3] are locally symmetric, which may occur in the family of metrics in theorem 11 if and only if $\gamma(y)$ is constant.

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